Lecture 21

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1 General properties of "area", "volume" and their generalizations

In this lecture we will give the general definition of the determinant of any square matrix.

On the last lecture we introduced the oriented area of the parallelogram. The main properties we used were the following:

1. $\operatorname{area}(a, b) = -\operatorname{area}(b, a)$ for any vectors a and b. From this condition it follows that $\operatorname{area}(a, a) = 0$ for any vector a. We'll prove it.

   Proof. Let $\operatorname{area}(a, b) = -\operatorname{area}(b, a)$ for all vectors a and b. Let $b = a$. Then $\operatorname{area}(a, a) = -\operatorname{area}(a, a)$ and so $\operatorname{area}(a, a) = 0$. □

2. $$\operatorname{area}(a_1 + a_2, b) = \operatorname{area}(a_1, b) + \operatorname{area}(a_2, b)$$
   $$\operatorname{area}(a, b_1 + b_2) = \operatorname{area}(a, b_1) + \operatorname{area}(a, b_2).$$

3. $\operatorname{area}(e_1, e_2) = 1$.

From these properties we defined the formula for the oriented area of the parallelogram. We should mention, that in order to derive this formula we needed ONLY these properties and nothing else!

By the same method we can define the oriented volume of the parallelepiped. It will satisfy the similar properties, and we can get similar formula for it, using ONLY these properties.

We will generalize this construction.

1.1 Generalization of the properties

The area was the function of 2 vectors in 2 dimensional space, and it was equal to the oriented area of the parallelogram on the 2-dimensional plane. Now we will consider the functions which takes n vectors from n-dimensional space as a parameters, and return a real number (For example, for oriented area we have $n = 2$, for oriented volume we have $n = 3$ — the function "volume" takes 3 vectors from the 3-dimensional space and returns a number — the value of the oriented volume).

So, let f be a function with n parameters, each of them is a vector from n-dimensional space.

Alternating functions. Function f is called **alternating** if for any number of vectors $a_1, a_2, \ldots, a_n$ we have

$$f(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) = -f(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_n),$$

i.e. if we interchange any 2 vectors, the function changes its sign.

For example, the corresponding property for "area" was $\operatorname{area}(a, b) = -\operatorname{area}(b, a)$. So, area is an alternating function.

From this property it directly follows that is 2 parameters of the function are equal, then the function is equal to 0:

$$f(a_1, \ldots, b, \ldots, b, \ldots, a_n) = 0.$$

It can be proved by the same manner as we did it for 2 dimensions (for "area").

Example 1.1. *Let $f(x, y) = x - y$. Then $f(x, y) = -f(y, x)$, so f is an alternating function. Moreover, we see that $f(x, x) = 0$. Actually, it's not a function which we will consider, but it is a nice example of alternating function.*

Multilinear functions. Function f is called **multilinear** if

$$f(a_1, \ldots, b + c, \ldots, a_n) = f(a_1, \ldots, b, \ldots, a_n) + f(a_1, \ldots, c, \ldots, a_n)$$

for any set of vectors $a_1, \ldots, b, c, \ldots, a_n$.

The corresponding property for "area" was $\operatorname{area}(a_1 + a_2, b) = \operatorname{area}(a_1, b) + \operatorname{area}(a_2, b)$ and $\operatorname{area}(a, b_1 + b_2) = \operatorname{area}(a, b_1) + \operatorname{area}(a, b_2)$. So we see that area is a multilinear function.

Normed functions. Function f is called **normed** is

$$f(e_1, e_2, \ldots, e_n) = \text{constant}.$$

For "area" the corresponding property was $\operatorname{area}(e_1, e_2) = 1$, so it was equal to the constant 1. Thus, area is a normed function.

2 Definition of the determinant

Now we can use these properties, and get the formula analogous to the formula of the oriented area on the plane. We will do it in the same way. Let f be a function, which takes n n-dimensional vectors as parameters, and it satisfies all the properties above, i.e. f is a multilinear alternating normed function.

Let $e_1, e_2, \dots, e_n$ be a basis in the n-dimensional vector space. Let's consider $f(a_1, a_2, \dots, a_n)$, where $a_1, a_2, \dots, a_n$ are n-dimensional vectors. We can represent each of these vectors as a linear combination of basic vectors:

$$\begin{aligned} a_1 &= a_{11}e_1 + a_{12}e_2 + \cdots + a_{1n}e_n \\ a_2 &= a_{21}e_1 + a_{22}e_2 + \cdots + a_{2n}e_n \\ &\dots \\ a_n &= a_{n1}e_1 + a_{n2}e_2 + \cdots + a_{nn}e_n \end{aligned}$$

Now we can do the following:

$$\begin{aligned} f(a_1, a_2, \dots, a_n) = f(&a_{11}e_1 + a_{12}e_2 + \cdots + a_{1n}e_n, \\ &a_{21}e_1 + a_{22}e_2 + \cdots + a_{2n}e_n, \\ &\dots \\ &a_{n1}e_1 + a_{n2}e_2 + \cdots + a_{nn}e_n). \end{aligned}$$

Now, performing the same calculations as we did for 2-dimensional case, we can derive the formula for f:

$$f(a_1, a_2, \dots, a_n) = c \cdot \sum_{\substack{\text{all permutations of} \\ n \text{ elements } \sigma}} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

Let's arrange coefficients of vectors a_{ij} and write them as matrix elements. So, let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \tag{1}$$

Then we can give the following definition. We want the function f to be normed such that $f(e_1, e_2, \dots, e_n) = 1$ (i.e $c = 1$).

Definition 2.1. *The **determinant** of the matrix A is defined by the following formula:*

$$f(a_1, a_2, \dots, a_n) = \sum_{\substack{\textit{all permutations of} \\ n \textit{ elements } \sigma}} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}. \tag{2}$$

We'll consider some examples in 2-dimensional space and in 3-dimensional space. We'll get the same formulae as we had on the last lecture.

2.1 Determinant of 2×2-matrix

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Let's use the formula (2) to get the determinant of A. There are $2! = 2$ different permutations of numbers 1 and 2. We'll consider both of them, take appropriate terms from the formula, and then add them up.

$\sigma = (12)$. There are no inversions here, so $\operatorname{sgn}(\sigma) = 1$. Then, $\sigma(1) = 1)$, and $\sigma(2) = 2$. So, the corresponding term from the formula is

$$1 \cdot a_{11}a_{22}$$

$\sigma = (21)$. There is 1 inversion here, so $\operatorname{sgn}(\sigma) = -1$. Then, $\sigma(1) = 2)$, and $\sigma(2) = 1$. So, the corresponding term from the formula is

$$-1 \cdot a_{12}a_{21}$$

Adding these 2 terms up, we get the formula for the determinant of A:

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

2.2 Determinant of 3×3-matrix

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Let's use the formula (2) to get the determinant of A. There are $3! = 6$ different permutations of numbers 1,2 and 3. We'll consider all of them, take appropriate terms from the formula, and then add them up.

$\sigma = (123)$. There are no inversions here, so $\operatorname{sgn}(\sigma) = 1$. Then, $\sigma(1) = 1$, $\sigma(2) = 2$ and $\sigma(3) = 3$. So, the corresponding term from the formula is

$$1 \cdot a_{11}a_{22}a_{33}$$

$\sigma = (132)$. There is 1 inversion here, so $\operatorname{sgn}(\sigma) = -1$. Then, $\sigma(1) = 1$, $\sigma(2) = 3$ and $\sigma(3) = 2$. So, the corresponding term from the formula is

$$-1 \cdot a_{11}a_{23}a_{32}$$

$\sigma = (213)$. There is 1 inversion here, so $\operatorname{sgn}(\sigma) = -1$. Then, $\sigma(1) = 2$, $\sigma(2) = 1$ and $\sigma(3) = 3$. So, the corresponding term from the formula is

$$-1 \cdot a_{12}a_{21}a_{33}$$

$\sigma = (231)$. There are 2 inversions here, so $\operatorname{sgn}(\sigma) = 1$. Then, $\sigma(1) = 2$, $\sigma(2) = 3$ and $\sigma(3) = 1$. So, the corresponding term from the formula is

$$1 \cdot a_{12}a_{23}a_{31}$$

$\sigma = (312)$. There are 2 inversions here, so $\operatorname{sgn}(\sigma) = 1$. Then, $\sigma(1) = 3$, $\sigma(2) = 1$ and $\sigma(3) = 2$. So, the corresponding term from the formula is

$$1 \cdot a_{13}a_{21}a_{32}$$

$\sigma = (321)$. There are 3 inversions here, so $\operatorname{sgn}(\sigma) = -1$. Then, $\sigma(1) = 3$, $\sigma(2) = 2$ and $\sigma(3) = 1$. So, the corresponding term from the formula is

$$-1 \cdot a_{13}a_{22}a_{31}$$

Adding these terms up, we get the formula for the determinant of A:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

Unfortunately, the number of terms grows very quickly with n, so it is almost impossible to compute the determinant of the 4×4-matrix using this formula. But there are many easy methods of computing the determinant. We will study them later.